Wall crossing and multi-centered black hole bound states

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J. Manschot, B. Pioline, A.S., arXiv:1011.1258; work in progress

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Plan

Two problems:

1. **Wall crossing in $\mathcal{N} = 2$ supersymmetric theories**

arXiv:1011.1258

2. **Bound state spectrum of multi-centered black holes in $\mathcal{N} = 2$ supergravity.**

work in progress

Wall crossing

1. Introduction

2. Our results and comparison with earlier results (Kontsevich-Soibelman formula)

3. ‘Derivation’

Even though we shall derive the result using black holes in $\mathcal{N} = 2$ supergravity, we expect this to be a general formula, valid also in $\mathcal{N} = 2$ supersymmetric gauge theories.

Introduction

Consider an $\mathcal{N} = 2$ supersymmetric string theory in D=4, *e.g.* type II string theory on a Calabi-Yau 3-fold.

Such a theory generically has a certain number n_v of vector multiplet fields.

$\Rightarrow$ there are $(n_v + 1)$ U(1) gauge fields.

A given state is characterized by $(n_v + 1)$ electric charges q_I and $(n_v + 1)$ magnetic charges p^I $(1 \le I \le n_v + 1)$.

$\gamma \equiv (q_I, p^I)$ takes values over some lattice Γ.

$$\langle \gamma, \gamma' \rangle \equiv (q_I p'^I - p^I q'_I)$$

Let t denote a point in the moduli space.

The BPS mass formula takes the form

$$m(\gamma, t) = |Z(\gamma, t)|, \qquad Z(\gamma, t) = q_I f^I(t) - p^I g_I(t)$$

for some known complex functions $f^I(t)$, $g_I(t)$.

We denote by $\Omega(\gamma, t)$ the index of BPS states carrying charge γ at the point t:

$$\Omega(\gamma, t) = \mathrm{Tr}'_\gamma((-1)^F)$$

Tr': removes the trace over the fermion zero modes associated with broken supersymmetries.

$\Omega(\gamma, t)$ is independent of t except for jumps across walls of marginal stability.

Let $\gamma_1, \gamma_2 \in \Gamma$ and take a codimension 1 subspace of the moduli space on which

$$\arg \mathbf{Z}(\gamma_1, \mathbf{t}) = \arg \mathbf{Z}(\gamma_2, \mathbf{t})$$

On this subspace (wall),

$$|\mathbf{Z}(\gamma_1 + \gamma_2, \mathbf{t})| = |\mathbf{Z}(\gamma_1, \mathbf{t})| + |\mathbf{Z}(\gamma_2, \mathbf{t})|$$

Thus a state of charge $(\gamma_1 + \gamma_2)$ is marginally unstable against decay into a pair of states carrying charges γ_1 and γ_2.

More generally a state carrying charge $(\mathbf{M}\gamma_1 + \mathbf{N}\gamma_2)$ becomes marginally unstable.

$\Rightarrow \Omega(\mathbf{M}\gamma_1 + \mathbf{N}\gamma_2, \mathrm{t})$ could jump as t crosses this wall.

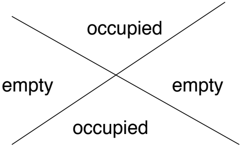
Goal: Compute the change in $\Omega(M\gamma_1 + N\gamma_2)$ across the wall.

Note: Only states carrying charges in the 2D sublattice spanned by γ_1 and γ_2 are relevant near this wall.

Assumption: After taking appropriate linear combinations, it is possible to choose the vectors γ_1 and γ_2 such that:

BPS states carrying charges $(M\gamma_1 + N\gamma_2)$ exist only for $M, N \geq 0$ or $M, N \leq 0$. Andriyash, Denef, Jafferis, Moore

$\Rightarrow$ **Define** $\tilde{\Gamma} \equiv \{m\gamma_1 + n\gamma_2 : m, n \geq 0, (m, n) \neq (0, 0)\}$



Consider two chambers in the moduli space separated by the wall.

$$c^+ : \quad \gamma_{12}\, \mathrm{Im}\left(Z^*_{\gamma_1} Z_{\gamma_2}\right) > 0, \qquad \gamma_{12} \equiv \langle \gamma_1, \gamma_2 \rangle$$

$$c^- : \quad \gamma_{12}\, \mathrm{Im}\left(Z^*_{\gamma_1} Z_{\gamma_2}\right) < 0$$

$\Omega^\pm(\gamma)$: index in these two chambers

We shall calculate Ω^- in terms of Ω^+

The results are best stated in terms of 'rational invariants'

... Joyce, Song; ...

$$\bar\Omega(\alpha) = \sum_{m|\alpha} m^{-2}\, \Omega(\alpha/m)$$

Results

Define: $\Delta\bar{\Omega}(\alpha) \equiv \bar{\Omega}^{-}(\alpha) - \bar{\Omega}^{+}(\alpha)$

Then

$$\Delta\bar{\Omega}(\alpha) = \sum_{\mathbf{n}\geq \mathbf{2}} \sum_{\substack{\{\alpha_{\mathbf{1}},\cdots\alpha_{\mathbf{n}}\in\tilde{\Gamma}\} \\ \alpha_{\mathbf{1}}+\cdots\alpha_{\mathbf{n}}=\alpha}} \frac{\mathbf{1}}{\mathbf{S}(\{\alpha_{\mathbf{k}}\})} \mathbf{g}(\alpha_{\mathbf{1}},\cdots\alpha_{\mathbf{n}})\,\bar{\Omega}^{+}(\alpha_{\mathbf{1}})\cdots\bar{\Omega}^{+}(\alpha_{\mathbf{n}})$$

If m_1 of the α_k's take the same value β_1, m_2 of the α_k's take the same value β_2 etc then

$$\mathbf{S}(\{\alpha_{\mathbf{k}}\}) = \mathbf{1}/\prod_{\mathbf{k}} \mathbf{m}_{\mathbf{k}}!$$

$g(\alpha_1, \cdots \alpha_n)$ will be given shortly.

Note: 'charge conservation rule' ($\alpha = \alpha_1 + \cdots \alpha_n$)

A generalization for the refined 'index'

KS, $\cdots$; Dimofte, Gukov

$$\Omega(\gamma, y) \equiv \mathrm{Tr}'_{\gamma}(-y)^{2J_3}$$

– not a protected index in string theory, but we can nevertheless calculate its jump across a wall of marginal stability.

$$\bar{\Omega}(\alpha, y) \equiv \sum_{m|\alpha} m^{-1} \frac{y - y^{-1}}{y^m - y^{-m}} \Omega(\alpha/m, y^m)$$

$$\Delta\bar{\Omega}(\alpha, y) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1, \cdots \alpha_n \in \tilde{\Gamma}\} \\ \alpha_1 + \cdots \alpha_n = \alpha}} \frac{1}{S(\{\alpha_k\})} g(\alpha_1, \cdots \alpha_n, y)\, \bar{\Omega}^+(\alpha_1, y) \cdots \bar{\Omega}^+(\alpha_n, y)$$

$g(\alpha_1, \cdots \alpha_n, y)$ will be given shortly.

$g(\alpha_1, \cdots \alpha_n, y) \to g(\alpha_1, \cdots \alpha_n)$ as $y \to 1$

We recover the earlier formulæ as $y \to 1$

Result for $g(\alpha_1, \ldots, \alpha_n, y)$ (for $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle > 0$ for $i < j$)

$$g(\alpha_1, \ldots, \alpha_n, y) = (-y)^{-1+n-\sum_{i<j} \alpha_{ij}} (y^2 - 1)^{1-n} \sum_{\text{partitions}} (-1)^{s-1} y^{2 \sum_{a \le b} \sum_{j<i} \alpha_{ji} m_i^{(a)} m_j^{(b)}}$$

The sum runs over all ordered partitions of $(\alpha_1 + \cdots + \alpha_n)$ into s vectors $\beta^{(a)}$ $(1 \le a \le s, 1 \le s \le n)$ such that

$$\beta^{(a)} = \sum_i m_i^{(a)} \alpha_i, \quad m_i^{(a)} = 0, 1 \quad \forall \quad i$$

$$\sum_{a=1}^{s} \beta^{(a)} = \alpha_1 + \cdots + \alpha_n$$

$$\left\langle \sum_{a=1}^{b} \beta^{(a)}, \alpha_1 + \cdots + \alpha_n \right\rangle > 0 \quad \forall \quad b \text{ with } 1 \le b \le s - 1$$

This wall crossing formula

– reproduces the semi-primitive wall crossing formula of Denef and Moore,

– agrees with the formulæ of Kontsevich and Soibelman and of Joyce and Song in all cases tested so far (up to n=5)

– the ‘charge conservation rules’ and the ‘identical particle rules’ agree with the KS formula after expressing the latter in terms of $\bar{\Omega}$.

However so far we do not have a proof of equivalence.

KS formula:

Gaiotto, Moore, Neitzke; Andriyash, Denef, Jafferis, Moore

Consider the algebra:

$$[\mathbf{e}_\gamma, \mathbf{e}_{\gamma'}] = \kappa(\langle\gamma, \gamma'\rangle, \mathbf{y})\ \mathbf{e}_{\gamma+\gamma'}\ ,$$

$$\kappa(\mathbf{x}, \mathbf{y}) \equiv \frac{(-\mathbf{y})^{\mathbf{x}} - (-\mathbf{y})^{-\mathbf{x}}}{\mathbf{y} - 1/\mathbf{y}}\ .$$

Define:

$$\mathbf{V}_\gamma^\pm = \exp[\bar{\Omega}^\pm(\gamma, \mathbf{y})\mathbf{e}_\gamma]$$

Then

$$\prod_{\substack{\mathbf{M}\geq 0, \mathbf{N}\geq 0,\\ \mathbf{M}/\mathbf{N}\downarrow}} \mathbf{V}^+_{\mathbf{M}\gamma_1+\mathbf{N}\gamma_2} = \prod_{\substack{\mathbf{M}\geq 0, \mathbf{N}\geq 0,\\ \mathbf{M}/\mathbf{N}\uparrow}} \mathbf{V}^-_{\mathbf{M}\gamma_1+\mathbf{N}\gamma_2}$$

– can be used to determine $\bar{\Omega}^-$ in terms of $\bar{\Omega}^+$.

‘Derivation’

Supergravity picture: Denef; Denef, Moore

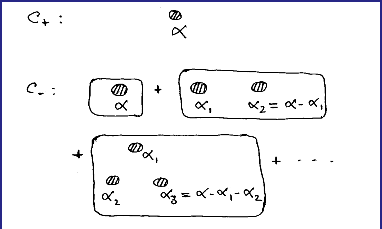
BPS states: Single centered black holes or multi-centered bound states of single centered black holes.

In c$^+$ the only configurations which contribute to $\Omega^+(\gamma)$ are single units (molecules) which remain immortal across the wall of marginal stability.

A molecule can contain a single BH of charge $\gamma \in \tilde{\Gamma}$ or bound states of multiple BH, each with charge lying outside $\tilde{\Gamma}$, but total charge $\gamma \in \tilde{\Gamma}$.

As we approach the wall the size of the molecule remains finite.

In c$^-$ the index $\Omega^-(\gamma)$ gets contribution from single molecules and also bound states of these molecules.



As we approach the wall from c$^-$ the intermolecular separation goes to ∞ and only single molecule states remain on the other side.

$\Rightarrow$ **Ω^+ describes the index for single molecules.**

Ω^- describes the index for single molecules + molecular bound states.

Naive guess:

$$\Delta\Omega(\alpha) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\cdots\alpha_n\in\tilde{\Gamma}\}\\ \alpha_1+\cdots\alpha_n=\alpha}} g(\alpha_1,\cdots\alpha_n)\,\Omega^+(\alpha_1)\cdots\Omega^+(\alpha_n)$$

$g(\alpha_1,\cdots\alpha_n)$: **Index of supersymmetric bound states of n centers with charges $\alpha_1,\cdots\alpha_n$, ignoring the internal contribution to the index from each center.**

Caveat: We need to take into account the effect of symmetrization for identical particles.

Example: Two identical bosonic centers, each with degeneracy Ω, will produce $\Omega(\Omega+1)/2$ states.

Combining this with non-trivial g is a complicated problem.

Strategy:

1. Argue that we can replace Bose/Fermi statistics by Boltzmann statistics if we replace Ω by $\bar{\Omega}$.

Then

$$\Delta\bar{\Omega}(\alpha) = \sum_{n\geq 2} \sum_{\substack{\{\alpha_1,\cdots\alpha_n \in \tilde{\Gamma}\} \\ \alpha_1+\cdots\alpha_n=\alpha}} \frac{1}{S(\{\alpha_k\})} g(\alpha_1,\cdots\alpha_n)\,\bar{\Omega}^+(\alpha_1)\cdots\bar{\Omega}^+(\alpha_n)$$

$S(\{\alpha_k\})$: Boltzmann factor ($m!$ for m identical particles)

2. Calculate $g(\alpha_1,\cdots\alpha_n)$ by computing the index associated with n-molecule quantum mechanics treating the different molecules as distinguishible (even if some of the α_k's are equal).

Bose/Fermi to Boltzmann

Consider a system of mutually non-interacting molecules carrying charges $\propto \gamma_0$, with total charge $k\gamma_0$.

e.g. m_s molecules with charge $s\gamma_0$, with $\sum_s s m_s = k$.

d_s: index of the molecule of charge $s\gamma_0$.

$$\text{Net index} \quad N_k = \sum_{\substack{\{m_s\} \\ \sum_s s m_s = k}} \prod_s \left[\frac{1}{m_s!} \frac{(d_s + m_s - 1)!}{(d_s - 1))!} \right]$$

$$N_k = \sum_{\substack{\{m_s\} \\ \sum_s s m_s = k}} \prod_s \left[\frac{1}{m_s!} \frac{(d_s + m_s - 1)!}{(d_s - 1))!} \right]$$

For bosons $d_s > 0$, and m_s identical bosons occupying d_s states produce a degeneracy of

$$d^{(B)} = d_s(d_s + 1) \cdots (d_s + m_s - 1)/m_s!$$

For fermions $d_s < 0$, and m_s fermions occupying $|d_s|$ states have total degeneracy

$$d^{(F)} = (|d_s|)(|d_s| - 1) \cdots (|d_s| - m_s + 1)/m_s!$$

and index

$$(-1)^{m_s} d^{(F)} = d_s(d_s + 1) \cdots (d_s + m_s - 1)/m_s!$$

Now consider a system of particles carrying charges $s\gamma_0$ for $s = 1, 2, \cdots$, obeying Boltzman statistics, and carrying index

$$\bar{d}_s = \sum_{m|s} m^{-1} d_{s/m}$$

Total index of a state carrying charge $k\gamma_0$ made out of these particles:

$$M_k = \sum_{\substack{\{m_s\} \\ \sum_s s m_s = k}} \prod_s \frac{1}{m_s!} (\bar{d}_s)^{m_s}$$

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$$\bar{d}_s = \sum_{m|s} m^{-1} d_{s/m}$$

$$N_k = \sum_{\substack{\{m_s\} \\ \sum_s s m_s = k}} \prod_s \left[\frac{1}{m_s!} \frac{(d_s + m_s - 1)!}{(d_s - 1))!} \right]$$

One can easily check that

$$\sum_k M_x x^k = \prod_n (1 - x^n)^{-d_n} = \sum_k N_k x^k$$

Thus

$$M_k = N_k$$

d_s gets contribution from orbital part $d_{orb}(s\gamma_0)$ and internal part $\Omega(s\gamma_0)$.

We shall argue that $d_{orb}(k\gamma_0) = k\, d_{orb}(\gamma_0)$

$$
\begin{aligned}
\Rightarrow \quad \bar{d}_s &= \sum_{m|s} m^{-1} d_{s/m} \\
&= \sum_{m|s} m^{-1} d_{orb}(s\gamma_0/m)\Omega(s\gamma_0/m) \\
&= \sum_{m|s} m^{-1} (1/m) d_{orb}(s\gamma_0)\Omega(s\gamma_0/m) \\
&= d_{orb}(s\gamma_0) \sum_{m|s} m^{-2}\Omega(s\gamma_0/m) = d_{orb}(s\gamma_0)\bar{\Omega}(s\gamma_0) \qquad (1)
\end{aligned}
$$

$\Rightarrow$ **we can use Boltzmann statistics provided $\Omega \to \bar{\Omega}$.**

Multi-centered black hole dynamics:

1. The centers move in the minimum of the potential.

If we fix the position of one then for every other center we have a 2 dimensional configuration space.

2. The centers interact via Lorentz force

– ignore interaction among parallel charges

3. Supersymmetry forces the system to be in the ground state (‘lowest Landau level’)

– the configuration space can be regarded as the phase space with a definite symplectic form

de Boer, El-Showk, Messamah, Van den Bleeken

Now compare two configurations:

1. k identical nearly coincident centers each with charge γ_0 moving in a background of other charges.

2. A center of charge $k\gamma_0$ moving in the same background.

Compare the phase space density associated with the two dimensional motion of $k\gamma_0$ vs the two dimensional motion of a single γ_0, keeping all the other charges fixed.

The phase space density of the particle of charge $k\gamma_0$ is k times that of the particle of charge γ_0.

$$d_{orb}(k\gamma_0) = k\, d_{orb}(\gamma_0)$$

We assume that the classical result is not corrected by quantum effects.

Our next goal is to compute g$(\alpha_1, \cdots \alpha_n, y)$.

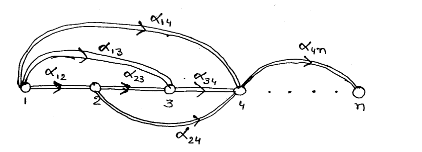
$\Rightarrow$ quantize multi-centered black hole solutions.

1. Indirect approach: relates the index associated with this quantum mechanics to that of a supersymmetric quiver quantum mechanics with Denef

a. n nodes each carrying a U(1) factor

b. $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle$ arrows from node i to node j

Result for this class of quivers is known and gives us back the formula for g$(\alpha_1, \cdots \alpha_n, y)$ given earlier. Reineke



2. Direct approach: Calculate the index by expressing it as an integral over the classical phase space and then evaluating this integral using localization techniques.

Duistermaat, Heckman

Result of direct approach (+ some guesswork based on known results for $n = 2, 3$)

$$g(\alpha_1, \cdots \alpha_n, y) = \mathrm{Tr}(-y)^{2J_3}$$

Of these $(-1)^{2J_3}$ takes the same value on all states:

de Boer, El-Showk, Messamah, Van den Bleeken

$$(-1)^{2J_3} = (-1)^{\sum_{i<j} \alpha_{ij} + n - 1}$$

y^{2J_3} is a slowly varying function of the phase space coordinates for $y \simeq 1$

$\Rightarrow$ $\mathrm{Tr}(y^{2J_3})$ can be evaluated as an integral over classical phase space

$$g_{cl}(\alpha_1, \cdots \alpha_n; y) = (-1)^{\sum_{i<i} \alpha_{ij} + n - 1} \int (y)^{2J_3}$$

$$g_{cl}(\alpha_1, \cdots \alpha_n; y) = (-1)^{\sum_{i<i} \alpha_{ij} + n - 1} \int (y)^{2J_3}$$

The phase space has an SU(2) rotation symmetry.

– can be used to express the integral as a sum over fixed points of rotation along z-axis

– collinear configuration of multi-centered black holes given by solutions to Denef

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\alpha_{ij}}{|z_{ij}|} = L \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij}, \qquad z_{ij} \equiv z_i - z_j, \qquad L > 0$$

Solutions labelled by permutations σ such that

$$z_{\sigma(i)} < z_{\sigma(j)} \qquad \text{for} \quad i < j$$

Then the phase space integral gives:

$$\begin{aligned} g_{cl}(\{\alpha_i\}, y) &= (-1)^{\sum_{i<j}\alpha_{ij}+n-1}(2\ln y)^{1-n} \\ &\quad \sum_{\text{permutations }\sigma} s(\sigma)\, y^{\sum_{i<j}\alpha_{\sigma(i)\sigma(j)}} \end{aligned}$$

The sum runs over only those permutations for which the solution exists.

$s(\sigma)$: a sign which can be determined

At $y = 1$ this agrees with the quiver quantum mechanics result in all cases tested.

$$g_{cl}(\{\alpha_i\}, y) = (-1)^{\sum_{i<j}\alpha_{ij}+n-1}(2\ln y)^{1-n} \sum_{\text{permutations } \sigma} s(\sigma)\, y^{\sum_{i<j}\alpha_{\sigma(i)\sigma(j)}}$$

– cannot be the correct quantum result for $\mathrm{Tr}(y^{2J_3})$ which must have $y \to e^{2\pi i} y$ symmetry.

– reflects the fact that classical phase space integral does not know about angular momentum quantization.

Taking clue from known results for $n = 2, 3$ we replace $2\ln y$ by $2\sinh\ln y = (y - y^{-1})$:

$$g(\{\alpha_i\}, y) = (-1)^{\sum_{i<j}\alpha_{ij}+n-1}(y - y^{-1})^{1-n} \sum_{\text{permutations } \sigma} s(\sigma)\, y^{\sum_{i<j}\alpha_{\sigma(i)\sigma(j)}}$$

$$
\begin{aligned}
\mathbf{g}(\{\alpha_{\mathbf{i}}\},\mathbf{y}) &= (-\mathbf{1})^{\sum_{\mathbf{i}<\mathbf{j}}\alpha_{\mathbf{ij}}+\mathbf{n}-\mathbf{1}}(\mathbf{y}-\mathbf{y}^{-\mathbf{1}})^{\mathbf{1}-\mathbf{n}} \\
&\quad \sum_{\text{permutations } \sigma} \mathbf{s}(\sigma)\,\mathbf{y}^{\sum_{\mathbf{i}<\mathbf{j}}\alpha_{\sigma(\mathbf{i})\sigma(\mathbf{j})}}
\end{aligned}
$$

– agrees with the quiver quantum mechanics results in all cases tested (up to n=5).

(also seems to work away from the wall of marginal stability where quiver quantum mechanics sometimes fails.)

Multi-centered black hole bound states

Consider an $\mathcal{N}=2$ supersymmetric string theory at a generic point of the moduli space.

For a given charge α the index receives contribution from single centered black holes as well as multi-centered black holes.

The index Ω for a single centered black hole can be computed in principle using quantum entropy function formalism. talk by Atish

Question: How can we use this information to compute the index associated with multi-centered black holes?

Naive guess: Use the same formula for the bound state spectrum as near the wall of marginal stability.

$$\sum_{n\geq 1} \sum_{\substack{\{\alpha_1,\cdots\alpha_n\in\Gamma\} \\ \alpha_1+\cdots\alpha_n=\alpha}} \frac{1}{S(\{\alpha_k\})} g(\alpha_1,\cdots\alpha_n,y)\,\bar\Omega(\alpha_1,y)\cdot\cdot\bar\Omega(\alpha_n,y)$$

$$g(\{\alpha_i\},y) = (-1)^{\sum_{i<j}\alpha_{ij}+n-1}(y-y^{-1})^{1-n} \sum_{\text{permutations } \sigma} s(\sigma)\, y^{\sum_{i<j}\alpha_{\sigma(i)\sigma(j)}}$$

Differences:

1. α_i's span the full lattice instead of 2D sublattice.

2. The Ω's now refer to index of single centered black holes rather than black hole molecules.

Caveats:

1. In this case the relevant equations are

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\alpha_{ij}}{|z_{ij}|} = c_i$$

c_i: constants which depend on moduli and charges

– necessary but not sufficient conditions for existence of collinear multi-black hole solutions.

For each such solution we must check that the corresponding metric is regular.

– a simple algebraic procedure.

2. For some charges, besides collinear solutions we also have ‘scaling solutions’ which are invariant under rotation about the z-axis.

– solutions where all the centers approach each other.

How do we evaluate the contribution from these additional fixed points?

Our proposal: use ‘minimal modification hypothesis’

1. First ignore the contribution from the scaling solutions and express the result for the index of bound states of centers carrying charges $\alpha_1, \cdots \alpha_n$ as sum of terms like

$$f(\cdots;y)\,\Omega(\alpha_{i_1}, y^{m_1})\cdots\Omega(\alpha_{i_k}, y^{m_k})$$

2. If $f(\cdots;y)$ is a finite linear combination of $y^{\pm m}$, i.e. if the denominators involving $(y^k - y^{-k})$ are cancelled by the numerator factors, then leave the term unchanged.

3. Otherwise add to f a function h such that

a. $(f+h)$ is a finite linear combination of $y^{\pm m}$.

b. h vanishes as $y \to \infty$.

This algorithm gives the correct result in all known cases.

Example:

Suppose we have a three centered configuration and suppose that the contribution from collinear solutions leads to the following result for $f(\alpha_1, \alpha_2, \alpha_3, y)$:

$$(-1)^l (y - y^{-1})^{-2} \left(y^l + y^{-l}\right), \quad l \equiv \sum_{i<j} \alpha_{ij}$$

– not a finite linear combination of $y^{\pm m}$.

Our prescription gives

$$(-1)^l (y - y^{-1})^{-2} \left(y^l + y^{-l} - 2\right) \quad \text{for } l \text{ even}$$

$$(-1)^l (y - y^{-1})^{-2} \left(y^l + y^{-l} - y - y^{-1}\right) \quad \text{for } l \text{ odd}$$

This gives a complete algorithm for computing the index for a given charge α in an $\mathcal{N} = 2$ supersymmetric string theory if we know the index Ω of single centered black holes.

This can then be compared with the microscopic results.